Lecture 22

There are two more isomorphism theorems. I am stating them here (and proving one of them). However, we won't study them in much detail. (They won't be in your syllabus for the exam. Yay!)

Theorem [Second Isomorphism Theorem]

If $K \leq G$ and $N \lhd G$, then $\dfrac{K}{K \cap N} \cong \dfrac{KN}{N}$.

Proof First note that if $K \leq G$ and $N \lhd G$ $\Rightarrow$ $KN \leq G$. $KN \neq \phi$ as $e \in KN$.

Let $a = k_1 n_1 \in KN$ for $k_1 \in K$, $n_1 \in N$.

$b = k_2 n_2 \in KN$ for $k_2 \in K$, $n_2 \in N$

Then $ab^{-1} = k_1 n_1 (k_2 n_2)^{-1} = k_1 \underbrace{n_1 n_2^{-1}}_{n_3} k_2^{-1}$

$$= k_1 n_3 k_2^{-1}$$

$$= \underbrace{k_1 k_2^{-1}}_{k_3} k_2 n_3 k_2^{-1}$$

$$= k_3 k_2 n_3 k_2^{-1}$$

But since $N \triangleleft G \Rightarrow k_2 n_3 k_2^{-1} \in N$ by the normal subgroup test. So

$ab^{-1} = k_3 n_4 \in KN \Rightarrow KN \leq G$.

Also, $N \triangleleft KN$. Let $k_1 n_1 \in KN$. Then

$$k_1 n_1 N = k_1 N \quad \text{— (1)}$$

also, since $N \triangleleft G$ and $k_1 \in G$ too $\Rightarrow k_1 N = N k_1$

$\Rightarrow$ $k_1 n_1 = n_2 k_1$ for some $n_2, n_3 \in N$

$\Rightarrow$ $N k_1 n_1 = N n_2 k_1 = N k_1$ — (2)

Since $N \triangleleft G \Rightarrow N k_1 = k_1 N \Rightarrow$ from (1) and (2) we get that $k_1 n_1 N = N k_1 n_1$

$\Rightarrow \quad N \trianglelefteq KN.$

Thus $\dfrac{KN}{N}$ is a group.

To prove $\dfrac{K}{K\cap N} \cong \dfrac{KN}{N}$, we'll use

Principle 3 :- Whenever you want to show that $\dfrac{G}{H} \cong \overline{G}$, try to find a surjective homomorphism $\varphi : G \to \overline{G}$ and show that $\ker\varphi = H$ and then apply the first Isomorphism Theorem.

So, define $\varphi : K \longrightarrow \dfrac{KN}{N}$ by

$$\varphi(k) = kN$$

φ is onto

for any $k_1 n_1 N \in \frac{KN}{N}$, $k_1 n_1 N = k_1 N$ and

so for $k_1 \in K$, $\varphi(k_1) = k_1 N$.

<u>φ is a homomorphism</u>

Let $k_1, k_2 \in K$. Then

$$\varphi(k_1 \cdot k_2) = k_1 k_2 N = k_1 N k_2 N = \varphi(k_1)\varphi(k_2)$$

So, φ is a homomorphism.

What is $\ker \varphi$?

$\ker \varphi = \{ k \in N \mid \varphi(k) = N \}$ as N is the identity in $\frac{KN}{N}$.

so, $\varphi(k) = kN = N \Rightarrow k \in N$. But $k \in K$

so, $k \in K \cap N \Rightarrow \ker(\varphi) = K \cap N$

and by the First Isomorphism Theorem (FIT)

$$\frac{K}{K \cap N} \cong \frac{KN}{N}.$$

$\boxtimes$

<u>Theorem</u> [Third Isomorphism Theorem]

If $M \lhd G$, $N \lhd G$ and $M \leq N$, then

$$\frac{\frac{G}{M}}{\frac{N}{M}} \cong \frac{G}{N}$$

i.e., G/M is a group as $M \lhd G$, $\frac{G}{N}$ is a group as $N \lhd G$. Since $M \leq N \Rightarrow M \lhd N \Rightarrow$ N/M is also a group. Finally, $\frac{N}{M} \lhd \frac{G}{M} \Rightarrow$

$\frac{G/M}{N/M}$ is also a group. The theorem says

the groups $\frac{G/M}{N/M} \cong \frac{G}{N}$.

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Finally, let's see one more application of FIT.

Theorem [Correspondence Theorem]

Let $\varphi : G \to \overline{G}$ be a surjective homomorphism. Consider the set $S = \{ H \leq G \mid \ker\varphi \leq H \}$ which is the set of all those subgroups in G which contain $\ker\varphi$. Consider the set $T = \{ \overline{H} \leq \overline{G} \}$ which is the set of subgroups of $\overline{G}$. Then

1) There is a bijection ψ b/w S and T.
2) If $H \in S$, i.e., $H \leq G$ and $\ker\varphi \leq H$, then $[G : H] = [\overline{G} : \psi(H)]$

So, the theorem is saying that every subgroup of G which contains $\ker\varphi$ corresponds to a unique subgroup of $\overline{G}$ and vice-versa, every subgroup of $\overline{G}$ corresponds to a subgroup of G which contains $\ker\varphi$.

So, the subgroup structure of $\overline{G}$ is same as the structure of subgroups of G containing $\ker\varphi$.

<u>Proof</u> We'll construct a bijection b/w S and T. Recall that if $\varphi : G \to \overline{G}$ is a homomorphism, then $\varphi(H) \leq \overline{G}$ for any $H \leq G$.

Define $\Psi : S \longrightarrow T$ by

$$\Psi(H) = \varphi(H)$$

i.e., take any subgroup in G and map it to its homomorphic image in $\overline{G}$.

Define $\overline{\Psi} : T \longrightarrow S$ by

$$\overline{\Psi}(\overline{H}) = \varphi^{-1}(\overline{H})$$

i.e., take any subgroup of $\overline{G}$ and map it to the inverse image of H under φ.

Recall that $\varphi^{-1}(\bar{H}) \leq G$.

We want to show that ψ and $\overline{\psi}$ are inver- -ses of each other and $\psi \circ \overline{\psi} = Id_T$ and $\overline{\psi} \circ \psi = Id_S$.

But even before that, why should $\overline{\psi}(\bar{H})$ lie in S, i.e, why should $\varphi^{-1}(\bar{H})$ contain $\ker \varphi$?

Well, $\varphi^{-1}(\bar{H}) = \{ g \in G \mid \varphi(g) \in \bar{H} \}$

since, $\bar{e} \in \bar{H} \implies \ker \varphi = \{ g \in G \mid \varphi(g) = \bar{e} \in \bar{H} \}$ is contained in $\varphi^{-1}(\bar{H})$ and hence $\overline{\psi}(\bar{H}) \in S$.

<u>Proof of 1)</u>

<u>Claim 1</u> $\psi \circ \overline{\psi} = Id_T$

Let $\bar{H} \in \overline{G} \implies \overline{\psi}(\bar{H}) = \varphi^{-1}(\bar{H})$

$\psi \circ \overline{\psi}(H) = \varphi(\varphi^{-1}(\bar{H}))$.

Note that, we cannot write $\varphi(\varphi^{-1}(\bar{H})) = \bar{H}$ as for example if $\varphi: G \to \bar{G}$ is the trivial homomorphism then $\varphi^{-1}(\bar{H}) = G$ and $\varphi(\varphi^{-1}(\bar{H})) = \varphi(G) = \bar{e} \neq \bar{H}$. In fact, this is the reason we are taking $S = \{ H \leq G,\ \ker\varphi \leq H \}$ as you'll see that a complication like just described won't occur if we choose subgroups from S.

<u>Want</u> :- $\varphi(\varphi^{-1}(\bar{H})) = \bar{H}$.

Let $\bar{h} \in \bar{H}$. Since φ is surjective $\Rightarrow \exists\ g \in G$ s.t. $\varphi(g) = \bar{h} \Rightarrow g \in \varphi^{-1}(\bar{H})$ and $\varphi(g) = \bar{h} \in \varphi(\varphi^{-1}(\bar{H})) \Rightarrow \bar{H} \subseteq \varphi(\varphi^{-1}(\bar{H}))$.

Conversely, if $h \in \varphi(\varphi^{-1}(\bar{H}))$

$\Rightarrow \exists\ g \in \varphi^{-1}(\bar{H})$ s.t. $h = \varphi(g)$

If $g \in \varphi^{-1}(\bar{H}) \Rightarrow \varphi(g) \in \bar{H} \Rightarrow h \in \bar{H}$

Thus Claim 1 is true.

Similarly, one can prove that $\bar{\psi} \circ \psi = Id_S$ and hence $\exists$ a bijection b/w S and T.

This proves 1) of the Theorem.

Now we'll prove part 2) i.e., if $H \in S$, then $[G : H] = [\bar{G} : \psi(H)]$.

It's enough to construct a bijection b/w the set of left cosets of H in G and the set of left cosets of $\psi(H)$ in $\bar{G}$.

Define $\Xi : \{gH : g \in G, H \in S\} \longrightarrow \{\bar{g}\bar{H} \mid \bar{H} \in T\}$

by $\Xi(gH) = \varphi(g)\varphi(H)$

Recall Principle 2 :- we must check that the map Ξ is well-defined as the domain is the set of cosets.

So, let $g_1 H = g_2 H \Rightarrow g_2^{-1} g_1 \in H$.

Now $\overline{\xi}(g_1 H) = \varphi(g_1)\varphi(H)$

$\overline{\xi}(g_2 H) = \varphi(g_2)\varphi(H)$

now $\varphi(g_2^{-1} g_1) = \varphi(g_2)^{-1}\varphi(g_1)$ [as φ is a homomorphism]

But if $g_2^{-1} g_1 \in H \Rightarrow$

$$\varphi(g_2^{-1} g_1) \in \varphi(H)$$

$$\Rightarrow \varphi(g_2^{-1} g_1)\varphi(H) = \varphi(H)$$

$$\Rightarrow \varphi(g_1)\varphi(H) = \varphi(g_2)\varphi(H)$$

Hence $\overline{\xi}(g_1 H) = \overline{\xi}(g_2 H) \Rightarrow \overline{\xi}$ is well-defined.

$\overline{\xi}$ is one-one

Let $\overline{\xi}(g_1 H) = \overline{\xi}(g_2 H)$

$$\Rightarrow \varphi(g_1)\varphi(H) = \varphi(g_2)\varphi(H)$$

$$\Rightarrow \varphi(g_2^{-1} g_1)\varphi(H) = \varphi(H)$$

$\Rightarrow \quad \varphi(g_2^{-1}g_1) \in \varphi(H)$

$\Rightarrow \quad g_2^{-1}g_1 \in H \Rightarrow g_2H = g_1H$ and $\bar{\xi}$ is one-one.

$\underline{\bar{\xi} \text{ is onto}}$.

Let $\bar{g}\varphi(H)$ be a coset of $\varphi(H)$. Since

φ is surjective $\Rightarrow \exists \; g \in G$ s.t. $\varphi(g) = \bar{g}$.

$\Rightarrow \quad \bar{\xi}(gH) = \varphi(g)\varphi(H) = \bar{g}\varphi(H)$.

$\Rightarrow \quad \bar{\xi}$ is onto.

Thus $\bar{\xi}$ is a bijection and hence we prove part 2).

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I know that this is neither the easiest nor the best proofs to see, so I do not expect you to learn this. However, it's important to understand the content of the theorem.

Recalls that if $\varphi : G \to \overline{G}$ is an isomorphism then $\ker\varphi = \{e\}$ and hence any $H \leq G$ contains $\ker\varphi$. So as a corollary of the correspondence theorem, we see that

<u>Corollary 1</u> Let $G \cong \overline{G}$. Then

1) The subgroup lattices of G and $\overline{G}$ are the same. [as $S = \{H \leq G\}$ and $T = \{\overline{H} \leq \overline{G}\}$].

2) $\forall\ k \in \mathbb{N}$, the number of subgroups in G and $\overline{G}$ of index k are the same.

Also, recall that if $N \trianglelefteq G$, then $N = \ker\varphi$ where $\varphi : G \to \dfrac{G}{N}$ is the natural homomorphism from $G \to G/N$. Then, the correspondence theorem gives

<u>Corollary 2</u> Let $N \trianglelefteq G$ and $\varphi : G \to \frac{G}{N}$ be the natural homomorphism. Then

(i) $\exists$ bijection b/w $\{H \leq G \mid N \leq H\}$ and $\{\bar{H} \leq \bar{G}\}$.

(ii) The bijection preserves the index of subgroups.

So corollary 2 is basically saying that the subgr-
-oup structure of the group $\frac{G}{N}$ is same as the

structure of subgroups of G containing N.

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